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On the Chiral Hubbard Model and the Chiral Kondo Lattice Model

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The integrability of the one-dimensional chiral Hubbard model is discussed in the limit of strong interaction, $U = \infty$. The system is shown to be integrable in the sense of the existence of an infinite number of constants of motion. The system is related to a chiral Kondo lattice model at strong interaction $J = +\infty$.

KEY WORDS: Hubbard model; Kondo lattice model; integrability.

The Hubbard model has been of considerable interest due to its possible relevance to high-temperature superconductivity. The one-dimensional Hubbard model with nearest neighbor hopping was solved by Lieb and Wu in 1968 with the help of the Bethe ansatz.[1-3, 2] It was shown that the system exhibits a metal–insulator phase transition at half-filling even for arbitrarily small interaction. Away from half-filling, the low-lying excitations of the system have been classified as Luttinger-liquid-like in the sense of Haldane. A few years ago, a one-dimensional $SU(2)$ Hubbard model with only relativistic right movers was introduced,[4] which reduces, at half-filling and large but finite on-site energy, to the $SU(2)$ Haldane–Shastry spin system with $1/r^2$ exchange interaction.[6, 3] Using the finite-size diagonalization result and the information provided by some special cases, an effective Hamiltonian was proposed which was used to provide the full energy spectrum and the thermodynamics for any on-site energy. With the help of the effective Hamiltonian, it was found that at $T = 0$ and half-filling,

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[2] See ref. 3 for a general reference book for researchers working in the field of exact solvabilities.

[3] Loosely speaking, the Haldane–Shastry spin system may be considered as a lattice version of the Calogero–Sutherland quantum system.[7]

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there exists a critical value U_c at which a metal–insulator phase transition occurs in the system.[4] It was conjectured that the system is completely integrable for any on-site energy. However, a proof for the conjectured energy spectrum and the thermodynamics, as well as the structure of the wavefunctions at finite U, is still unknown. The integrability of this Hubbard model remains an open problem.

In the strong-interaction limit $U=\infty$, it was discovered that the Gutzwiller–Jastrow product wavefunctions are eigenstates of the chiral Hubbard model, both in the $SU(2)$ case and in the $SU(N)$ case.[5,4] In fact all eigenstates can be expressed in terms of more generalized Jastrow product wavefunctions. Furthermore, the $SU(2)$ energy spectrum[5] is the same as conjectured by Gebhard and Ruckenstein in their original work.[4] In this work, we first discuss the integrability of the system in the limit where the interaction between the electrons is infinitely strong, $U=\infty$. Using a simple argument, we shall exhibit an infinite number of constants of motion, showing that the system is integrable.

In the second part of the paper, we consider a one-dimensional chiral Kondo lattice. The conduction band has only right-moving electrons, and the electrons interact with each localized impurity moment through an exchange interaction. We identify the chiral Kondo lattice at $J=+\infty$ with the Hubbard model which has been previously studied. With this identification, the full energy spectrum, the wavefunctions, the thermodynamics, and the integrability of the system can be obtained for the Kondo lattice in this limit. In particular, various correlation functions between the electrons and the impurity spins can be computed exactly for this system.

The chiral Hubbard model Hamiltonian is defined on a one-dimensional lattice of length L:

$$H=\sum_{1\leqslant i\neq j\leqslant L}\sum_{\sigma=1}^{N}(t_{ij}c_{i\sigma}^{\dagger}c_{j\sigma})+U\sum_{1\leqslant i\leqslant L}\sum_{\sigma\neq\sigma'}n_{i\sigma}n_{i\sigma'} \tag{1}$$

where the hopping matrix element is given by

$$t_{mn}=(-it)(-1)^{(m-n)}\{(L/\pi)\sin[\pi(m-n)/L]\}^{-1}$$

For this $SU(N)$ system, the spin of the electrons can take values from 1 to N. In the case of $SU(2)$, it is the Hamiltonian introduced by Gebhard and Ruckenstein.[4] In the strong-interaction limit $U=\infty$, at each site there is at most one electron; the number of holes N_h and the number of

[4] Correlation functions have been computed by Forrester[15] for the eigenstates of this $1/r$ Hubbard model in the strong-interaction limit $U=\infty$.

electrons N_e on the lattice are conserved quantities. To rewrite the Hamiltonian in more convenient form, we perform the following unitary transformation T:

$$\begin{aligned} c^\dagger_{x\sigma} &\to (-1)^x e^{-\pi i x/L} c^\dagger_{x\sigma} \\ c_{x\sigma} &\to (-1)^x e^{\pi i x/L} c_{x\sigma} \end{aligned} \tag{2}$$

under which the original Hamiltonian becomes

$$H \to \bar{H} = 2tP_G \left[\sum_{1 \leqslant i \neq j \leqslant L} \sum_{\sigma} \frac{z_j}{(z_i - z_j)} c^\dagger_{i\sigma} c_{j\sigma} \right] P_G \frac{\pi}{L} \tag{3}$$

with P_G, the Gutzwiller projector, making sure that there are no double or multiple occupancies, while $z_x = e^{2\pi i x/L}$, with $x = 1, 2, \ldots, L$.

In the Hilbert space of no double or multiple occupancies, the electron fields can be rewritten with the superalgebra representations,

$$\begin{aligned} P_G(i)\, c^\dagger_{i\sigma} P_G(i) &= f^\dagger_{i\sigma} b_i \\ P_G(i)\, c_{i\sigma} P_G(i) &= b^\dagger_i f_{i\sigma} \end{aligned} \tag{4}$$

where the f fields are fermions, the b fields are bosons, with the constraint $\sum_\sigma f^\dagger_{i\sigma} f_{i\sigma} + b^\dagger_i b_i = 1$. Here $P_G(i)$ is the Gutzwiller projector operator on the site i for the electron operators c and $c^\dagger$. Any state vector of the Hilbert space can be written as

$$|\phi\rangle = \sum_{\{x\sigma\},\{y\}} \phi(\{x\sigma\},\{y\}) \prod_{i=1}^{N_e} f^\dagger_{x_i\sigma_i} \prod_{j=1}^{N_h} b^\dagger_{y_j} |0\rangle \tag{5}$$

where the amplitude ϕ is symmetric in the coordinates $\{y\}$ of the b bosons, and antisymmetric when exchanging the spin and positions $x_i\sigma_i$, $x_j\sigma_j$ of two f fermions. Here, $(q_1, q_2, \ldots, q_L) = (x_1, \ldots, x_{N_e}, y_1, \ldots, y_{N_h})$ span the full chain. The eigenenergy equation of the system can then be written in the first-quantized form as follows:

$$\begin{aligned} &\frac{2t\pi}{L} \left[\sum_{i=1}^{N_e} \sum_{j=1\,(\neq i)}^{L} \frac{Z_j}{Z_i - Z_j} M_{ij} - \frac{1}{2} \sum_{1 \leqslant i \neq j \leqslant N_e} P^\sigma_{ij} \right] \phi(\{q\},\{\sigma\}) \\ &\quad = \dot{E}\phi(\{q\},\{\sigma\}) \end{aligned} \tag{6}$$

where $Z_i = \exp(2\pi i q_i/L)$, the operator M_{ij} exchanges the position variables q_i and q_j, and the operator P^σ_{ij} exchanges the f-fermion spin variables σ_i and σ_j. These two operators commute with each other, as they act on different groups of variables of the wavefunction.

Following the ideas of refs. 12, we define generalized momentum operators Π_i, with $i=1, 2, ..., N_e$,

$$\Pi_i = \sum_{j=1\ (\neq i)}^{L} V_{ij} M_{ij} \tag{7}$$

with $V_{ij}=Z_j/(Z_i-Z_j)$. With this special form of V_{ij}, the generalized momentum operators satisfy the commutation relation

$$[\Pi_i, \Pi_j] = M_{ij}\Pi_i - \Pi_i M_{ij} \tag{8}$$

where $1 \leqslant i \neq j \leqslant N_e$. We then introduce the following Hermitian operators:

$$A_n = \sum_{s=1}^{N_e} \Pi_s^n \tag{9}$$

where $n=0, 1, 2,..., \infty$, and the sum s is over the electrons, i.e., from 1 to N_e. In particular, the Hamiltonian is given by

$$\hat{H} = \frac{2\pi t}{L}\left(A_1 - \frac{1}{2}\sum_{1 \leqslant i \neq j \leqslant N_e} P_{ij}^{\sigma}\right) \tag{10}$$

Using the commutation relations (8), it can be shown that all the operators A_n commute with each other. Furthermore, the action of A_n on some amplitude ϕ does not change the symmetry properties, i.e., the resultant wavefunctions remain symmetric in the b-boson positions and antisymmetric when exchanging any pair of the f-fermion positions and spins simultaneously. It is straightforward to prove the following relations:

$$\begin{aligned} M_{ij}[A_n\phi] &= [A_n\phi], & N_e+1 \leqslant i \neq j \leqslant L \\ M_{ij}P_{ij}^{\sigma}[A_n\phi] &= (-1)[A_n\phi], & 1 \leqslant i \neq j \leqslant N_e \end{aligned} \tag{11}$$

Since all the operators A_n commute with the Hamiltonian given by Eq. (10), we thus have an infinite set of conserved physical quantities of the system, showing that the system is indeed completely integrable. With these A_n we can construct corresponding quantities written in second-quantized language, which commute among themselves and with the Hamiltonian $\bar{H}$. Carrying out the unitary transformation T^{-1}, it is straightforward to convert them so that the resultant quantities are constants of the original Hamiltonian H.

One interesting observation is that the mutually commuting Hermitian quantities A_n are also the invariants of the long-range supersymmetric t-J

model,[8,5] i.e., $[A_n, H_{t\text{-}J}]=0$. As noted previously, the physical quantities $I_n=\sum_{i=1}^{L} \Pi_i^n$, with $n=0, 1, 2,...$, commute with each other and with the supersymmetric t-J model Hamiltonian. The two families of quantities $\{I_n\}$ and $\{A_n\}$ are independent of each other, in the sense that we cannot write any member of one group in terms of a linear combination of members of the other group. Furthermore, by explicit computation, one can show that they do not commute, e.g., $[A_n, I_m] \neq 0$. Therefore, it is clear that the previous family of conserved quantities $\{I_n\}$, although providing a proof of the integrability of the t-J model, does not exhaust all the constants of motion. The two infinite symmetries of the system do not commute with each other.[6] It would be very interesting to find a larger group of mutually commuting constants of motion. This is actually one of the most fundamental questions encountered in study of quantum integrable systems. That is, when one has found an infinite number of simultaneous constants of motion, showing that the system is integrable, it is still not certain that this infinite set contain *all* possible simultaneous constants of motion of the system.

In the following, we will show how the results of the chiral Hubbard model may be generalized to a one-dimensional Kondo lattice model. The Kondo lattice model has been an interesting model for the study of heavy fermion systems.[11] In this model, the system has an array of localized impurity moments, and the conduction electrons interact with the local moments through spin exchange. In general, the conduction band is best described by the tight-banding picture in which we have both right and left movers. However, in the following we assume that the electrons propagate in only one direction. This chiral Kondo lattice model is defined on a one-dimensional lattice, with the Hamiltonian

$$H=\sum_{k} \sum_{\sigma=\uparrow,\downarrow} e(k)\, c_{k\sigma}^{\dagger} c_{k\sigma} + J \sum_{i=1}^{L} c_{i\alpha}^{\dagger} \frac{\boldsymbol{\sigma}_{\alpha\beta}}{2} c_{i\beta} \cdot \mathbf{S}_f(i) \tag{12}$$

where the conduction band spectrum is $e(k)=-tk$, with $k=2\pi K/L$, $-(L-1)/2 \leqslant K \leqslant (L-1)/2$, in the momentum space. J is the coupling constant between the local impurity moments and the conducting electrons. The local moments are described by the spin-1/2 operators, that is, $[S_f^x(k), S_f^y(k)]=iS_f^z(k)$ (plus two other commutation relations obtained by the cyclic permutations of x, y, z), with the relation $\mathbf{S}_f^2(k)=3/4$, for all

[5] The constants of motion of the long-range t-J models were also discussed recently.[9] The spectrum of the t-J model on uniform chain is given in Ref. 10.

[6] A similar situation might happen in the conventional one-dimensional Hubbard model of nearest neighbor hopping and finite on-site energy U (E. H. Lieb, private communication).

the sites $k = 1, 2, ..., L$. The Hamiltonian may also be written in the following way:

$$H = \sum_{1 \leqslant i \neq j \leqslant L} \sum_{\sigma = \uparrow, \downarrow} t_{ij} c_{i\sigma}^{\dagger} c_{j\sigma} + J \sum_{i=1}^{L} c_{i\alpha}^{\dagger} \frac{\boldsymbol{\sigma}_{\alpha\beta}}{2} c_{i\beta} \cdot \mathbf{S}_f(i) \tag{13}$$

where $t_{mn} = (-it)(-1)^{(m-n)} \{(L/\pi) \sin[\pi(m-n)/L]\}^{-1}$.

When the interaction of the electron and the impurity is very strong, i.e., $J = +\infty$, we can map the system onto the above chiral Hubbard model with infinite repulsion. Indeed, when there are N_e electrons on the lattice L, with $N_e \leqslant L$, then each electron will attempt to form a singlet with the impurity spin at each site, to lower the energy of the system as much as possible, and some unpaired impurity spins are left over on the lattice. The Hilbert space at each site can be either an unpaired impurity spin or a singlet of electron–impurity bound state. Due to the hopping of the conduction electrons, the singlets can hop on the lattice. In this case, the basis vectors can be written as

$$|\alpha\rangle = 2^{-N_e/2} \left[\prod_{i=1}^{N_e} (1 - P_{\gamma_i \beta_i}) \right] c_{x_1 \gamma_1}^{\dagger} c_{x_2 \gamma_2}^{\dagger} \cdots c_{x_{N_e} \gamma_{N_e}}^{\dagger} |0\rangle$$
$$\otimes |\sigma_1, \sigma_2, ..., \beta_1, ..., \beta_2, ..., \sigma_{L-N_e}\rangle \tag{14}$$

where the singlets are located at positions $\{x\} = (x_1 < x_2 < \cdots < x_{N_e})$, and the unpaired impurity spins $(\sigma_1, \sigma_2, ..., \sigma_{L-N_e})$ are positioned at sites $\{y\} = (y_1 < y_2 < \cdots < y_{L-N_e})$. Here, the operator $P_{\gamma_i \beta_i}$ permutes the spin indices γ_i and β_i, to form a singlet of electron and impurity at site x_i. With P the projector onto this subspace, the Hamiltonian takes the form

$$\tilde{H} = PHP = PTP + c \tag{15}$$

where T is the kinetic energy of the conduction electrons; the infinite constant $c = (-J/4) N_e$ only shifts the origin of the energy of the system, a reference energy which is unimportant physically. In the space where the z component of the total spin is fixed, that is, $S_z = M$, the number of unpaired up-spin impurities is $A = M + (L - N_e)/2$, and the number of unpaired down-spin impurities is $B = -M + (L - N_e)/2$. Here $C_L^{N_e} \times C_{L-N_e}^{A}$ is the size of the Hilbert space. Any eigenstate of the Hamiltonian $H_1 = PTP$ can be written as a linear combination of the basis vectors,

$$|\phi\rangle = \sum_{\alpha} C(\alpha) |\alpha\rangle \tag{16}$$

We can identify the singlets as spinless fermions, and the unpaired

impurities as hard-core spin-1/2 bosons hopping on the lattice. Let us consider a system described by the following Hamiltonian:

$$h=(1/2)\sum_{i\neq j,\,\sigma} P_G(t_{ij}b^\dagger_{j\sigma}b_{i\sigma}\,g^\dagger_i\,g_j)\,P_G \tag{17}$$

where the b fields are bosonic, g fields are fermionic, b fields commute with the g fields, and the Gutzwiller projector is

$$P_G=\prod_{i=1}^{L}[\delta_{1,\,g^\dagger_i g_i+\sum_{\sigma=\uparrow,\downarrow} b^\dagger_{i\sigma}b_{i\sigma}}]$$

The basis vectors may be represented as follows:

$$|\bar{\alpha}\rangle = g^\dagger_{x_1}\,g^\dagger_{x_2}\cdots g^\dagger_{x_{N_e}}b^\dagger_{y_1\sigma_1}b^\dagger_{y_2\sigma_2}\cdots b^\dagger_{y_{L-N_e}\sigma_{L-N_e}}\,|0\rangle \tag{18}$$

One can show that the systems described by H_1 and h are isomorphic to each other, by verifying the following matrix elements:

$$\langle\beta|\,H_1\,|\alpha\rangle=\langle\bar{\beta}|\,h\,|\bar{\alpha}\rangle \tag{19}$$

where there is a one-to-one correspondence $|\alpha\rangle\leftrightarrow|\bar{\alpha}\rangle$ for the basis vectors. The Hamiltonian h is equivalent to the following Hamiltonian:

$$h=(1/2)\sum_{i\neq j,\,\sigma}(-t_{ji})\,P_F F^\dagger_{i\sigma}F_{j\sigma}P_F \tag{20}$$

where $P_F=\prod_{i=1}^{L}P_F(i)$, $P_F(i)=(1-F^\dagger_{i\uparrow}F_{i\uparrow}F^\dagger_{i\downarrow}F_{i\downarrow})$, and $L-N_e$ is the number of the F fermions on the lattice.

With the above identification, we have mapped the chiral Kondo lattice model onto the chiral Hubbard model with strong repulsion. Therefore, an infinite number of mutually commuting invariants can be obtained for the Kondo Lattice model. The wavefunctions and the thermodynamics of the system may be read off from previous results.[5] Any state vectors can be written as

$$|\phi\rangle=\sum_{\{X\},\,\{Y\}}\Phi(\{X\},\{Y\})\prod_{i=1}^{\bar{Q}}F_{Y_i\uparrow}\prod_{j=1}^{B}F^\dagger_{X_j\downarrow}F_{X_j\uparrow}\,|P\rangle \tag{21}$$

where $|P\rangle=\prod_{i=1}^{L}F^\dagger_{i\uparrow}\,|0\rangle$, $\bar{Q}=N_e$ is the number of g fermions, $B=-M+(L-N_e)/2$ is the number of down-spin b bosons, and the amplitude Φ is antisymmetric in the positions $\{Y\}$ and symmetric in the positions $\{X\}$. The following Jastrow wavefunctions are eigenstates of the Hamiltonian:

$$\Phi(\{X\}, \{Y\}) = \exp\frac{2\pi i}{L}\left(m_s \sum_i X_i + m_h \sum_j Y_j\right) \times \prod_{i<j} d^2(X_i - X_j) \prod_{i<j} d(Y_i - Y_j) \prod_{i,j} d(X_i - X_j) \tag{22}$$

with $d(n) = \sin(\pi n/L)$. The quantum numbers m_s, m_h are integers or half-integers, which ensures the periodic boundary conditions, satisfying the following constraints:

$$\begin{aligned} |m_h| &\leqslant L/2 - (B + \bar{Q})/2 \\ |m_h - m_s - L/2| &\leqslant L/2 - (A + \bar{Q})/2 \end{aligned} \tag{23}$$

with the eigenenergies given by

$$E(m_s, m_h) = -(2\pi t/L)[2m_h - m_s + L/2]\ \bar{Q}(1/2) \tag{24}$$

The full spectrum of the system takes the following form:

$$E = -(2\pi t/L)\left[\sum_{i=1}^{\bar{Q}} n_i + \sum_{\mu=1}^{\bar{Q}} m_\mu\right](1/2) \tag{25}$$

Here the integers (or half-integers) satisfy the conditions $|n_i| \leqslant L/2 - (A + \bar{Q})/2$ and $|m_\mu| \leqslant L/2 - (B + \bar{Q})/2$, where $n_i \leqslant n_{i+1}$ and $m_\mu \leqslant m_{\mu+1}$. This result shows that the spectrum is invariant when changing the sign of t.

The Jastrow product wavefunctions of the unpaired impurity spins and the singlets are typical RVB-type wavefunctions. Various correlation functions of the impurity spins and the singlets can be computed *exactly*, by trivially generalizing Forrester's work to this case. It should be remarked that the faraway unpaired impurity spins are also strongly correlated with each other, because only right movers exist in the conduction band. At half-filling, the system is obviously an insulator, since each electron forms a singlet at each site and the singlets cannot hop from one site to another.

For this chiral Kondo lattice model, the conduction electrons move only in one direction. We can anticipate many physical properties for the system even at finite coupling constant J. Away from half-filling, one would expect the system to be in a metallic state. At half-filling and for sufficiently large J, the system is expected to be insulating. In this model, the chirality of the conduction band will not prevent the system from becoming insulating, unlike in some other situations, such as in the edges of the fractional

quantum Hall effect, where the chiral Luttinger liquid will not become localized under any randomness, due to lack of backscattering of the quasiparticle.[13] In our case, although th electrons are moving in only one direction, the mechanism for localization is very different. The electron always feels the exchange interaction of the impurity spin, through the spin exchange interaction. For J large enough, each electron will attempt to form a localized singlet with each impurity spin; therefore, at half-filling, to transfer one electron from one site to another would break two singlets, causing a charge gap of $O(J)$, and the system would be in an insulating state. At $J=0$, the system is a simple Fermi liquid. One would thus expect that there exists a critical coupling J_c where the system exhibits a metal–insulator phase transition at half-filling. An interesting question is whether any infinitesimal small J would drive the conducting band to an insulating state at half-filling, i.e., $J_c=0^+$. If $J_c\neq 0^+$, one would expect $J_c\sim |t|$ by dimensional analysis, and the system is metallic for $0<J<J_c$, while it becomes insulating for $J_c<J$. The metal–insulator phase transition would also occur at half-filling when changing $|J|$ for ferromagnetic interaction between the conduction electrons and the impurity spins. Further work is necessary to locate the critical coupling J_c. It might also be interesting to see whether the model at finite J belongs to Jastrow-integrable type.

In summary, we have obtained an infinite number of constants of motion for the one-dimensional chiral Hubbard model in the strong-interaction limit $U=\infty$. We have also shown that this model is equivalent to the one-dimensional chiral Kondo lattice model at $J=+\infty$. It seems that the integrability condition might be investigated for finite on-site energy, using a similar approach. However, we have not succeeded in doing so for finite on-site energy. It has also seemed that the finite-J chiral Kondo lattice model is very probably integrable, as the conduction electrons move in one direction, only exchanging spins with the local moments. In the continuum limit, loosely speaking, the many-particle scattering matrices are very probably factorized into two-body ones with a continuum of relativistic electrons, and the situation of scattering matrices of the electrons off one impurity might be similar to the S-matrices of the solution of Andrei *et al.*[14] Further work, either numerically or analytically, is necessary for evidence of its solvability.

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